Nucleosynthesis Without Computer

V. Mukhanov[1]

Complete analytical derivation of the time evolution and final abundances of the light elements (up to ^{7}Be) formed in the big-bang nucleosynthesis is presented. This highlights an interesting physics taking place during the formation of light elements in the early universe.

1. INTRODUCTION

The most occurrent chemical element in the universe is hydrogen. It constitutes nearly three quarter of all baryonic matter. The next mostly wide spread element Helium-4, constitutes about 25%. The other light elements and the metals occur very rare. Very simple arguments lead to the conclusion that it is very unlikely that ^{4}He, deuterium (D), and other light elements could be burned in the stars (see, for instance, Kolb and Turner (1993), Weinberg (1972)). Therefore the only sensible explanation of their abundance is that they were produced in the very early universe. It is clear that the essential amount of the helium could not be formed before the temperature dropped below its binding energy ~ 28 MeV and one can expect that the Big-Bang nucleosynthesis (BBN) took place when the temperature was not very different from $\sim$MeV, that is, somewhere in between seconds and minutes after the Big-Bang. Therefore BBN, being based on the well understood physics, offers the possibility of reliable probe of the early universe (see for instance, Kolb and Turner (1993), Malaney and Mathews (1993), Olive *et al.* (2000), Sarkar (1996), Schramm and Turner (1998), Walker *et al.* (1991), Weinberg (1972) and references cited therein). The amount of the produced elements depend on the basic cosmological parameters and is very sensitive to the baryon density. The measured abundances combined with the CMB temperature fluctuation measurements provide us a unique opportunity to verify the reliability of the "standard model" of the universe evolution (Bennett *et al.*, 2003; Fields and Sarkar, 2002).

[1] LMU, Sektion Physik, Theresienstr. 37, 80333 Muenchen, Germany; e-mail:mukhanov@theorie. phsik.uni-muenchen.de.

The element abundances, are usually calculated using computer codes (for instance, publically available (www-thphys.physics.ox.ac.uk/users/SubirSarkar/bbn.htlm) Wagoner code (Kawano, 1992; Wagoner, 1973; Wagoner *et al.*, 1967) and the abundances are presented as the function of the baryon density. To understand the dependence of the element abundances on the cosmological parameters the semianalytical and analytical description of BBN proved to be very useful (Berstein *et al.*, 1989; Esmailzadeh *et al.*, 1991). In this paper I develop simple quasi-equilibrium analytical approach which allows to derive the final abundances of all light elements up to Berilium-7 without using any computer codes. The accuracy of the results is very good for ^{4}He, good for D, and reasonably good for the other elements. I obtain analytical (not fitting) formulae describing the dependence of the abundances on the cosmological parameters and trace the time evolution of the element abundances before their freeze-out. This highlights an interesting and reach physics taking place during nucleosynthesis and allows to understand the physical reasons for the dependence of the abundances on parameters without practicing with computer codes.

2. FREEZE-OUT

The amount of helium produced depends on the availability of the neutrons at the time when the helium is formed. In turn, the neutron concentration is determined by the weak interactions which ensure the chemical equilibrium between the neutrons and protons at very early time. The weak interactions become inefficient when the temperature drops below few MeV. Around this time the neutrons chemically decouple from the protons and after that the ratio of their concentrations "freeze out."[2] The nuclear reactions take place after that. Therefore, first we need to calculate the "freeze out" concentration of the neutrons.

The main processes responsible for the chemical equilibrium between protons and neutrons in the early universe are the weak interaction reactions:

$$n + \nu \rightleftarrows p + e^-, \quad n + e^+ \rightleftarrows p + \bar{\nu} \tag{1}$$
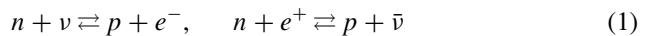
Here ν always means the electron-neutrino. To calculate the rate of these reactions one can use the Fermi theory according to which the matrix element characterizing 4-fermion interaction (1) is equal:

$$|\mathcal{M}|^2 = 16(1 + 3g_A^2)G_F^2(p_n \cdot p_\nu)(p_p \cdot p_e), \tag{2}$$

where $G_F = \pi \alpha_w / \sqrt{2} M_W^2 \simeq 1.17 \times 10^{-5}/\text{GeV}^2$ is the Fermi coupling constant, $g_A \simeq 1.26$ is the correction to the axial vector "weak charge" of the

---

[2] The above statement is, of course, literally true only if one neglects the neutron decay.

nucleon,[3] and $(p_i \cdot p_j)$ are the scalar products of appropriate 4-momenta entering the vertex.[4] Considering the process $a + b \to c + d$ of type (1), we get the following expression for the differential cross-section of this interaction:

$$\frac{d\sigma_{ab}}{d\Omega} = \frac{1}{(8\pi)^2} \frac{|\mathcal{M}|^2}{(p_a + p_b)^2} \left(\frac{(p_c \cdot p_d)^2 - m_c^2 m_d^2}{(p_a \cdot p_b)^2 - m_a^2 m_b^2} \right)^{1/2}, \tag{3}$$

where the integration over the phase space of c, d-particles has been performed. This expression is manifestly Lorentz-invariant and can be used in any coordinate frame. Note, that the 4-momenta of the produced particles are related to the 4-momenta of the colliding particles via the conservation law: $p_c + p_d = p_a + p_b$. Let us now consider the particular reaction $n + \nu \to p + e^-$ at the temperatures around few MeV and below. In such a case the nucleons are nonrelativistic; hence

$$(p_n + p_\nu)^2 \simeq m_n^2; \quad (p_n \cdot p_\nu) = m_p \epsilon_\nu;$$

$$\sqrt{(p_p \cdot p_e)^2 - m_p^2 m_e^2} \simeq m_p \epsilon_e \sqrt{1 - (m_e/\epsilon_e)^2} = m_p \epsilon_e v_e \tag{4}$$

where ϵ_ν is the energy of the incoming neutrino and $\epsilon_e \simeq \epsilon_\nu + Q$ is the energy of the outgoing electron. The energy $Q = 1.293$ MeV, is released when the neutron "is converted" into proton. The formula (3) is directly applicable only in empty space. However, at the temperatures above 0.5 MeV there still present many $e^\pm$-pairs and the possible final states for the electron are partially occupied. Because of the Pauli exclusion principle it reduces the appropriate cross-section by the factor $(1 - n_{\epsilon_e}) = (1 + \exp(-\epsilon_e/T))^{-1}$. Taking this into account and substituting (4) into (2), (3) one gets:

$$\sigma_{n\nu} \simeq \frac{1 + 3g_A^2}{\pi} G_F^2 \epsilon_e^2 v_e (1 + \exp(-\epsilon_e/T))^{-1}, \tag{5}$$

where we have neglected the chemical potential of the electrons. Note that the concentration of the nucleons is negligible compared to the concentration of the light particles at this time and therefore the spectrum of the light particles is practically not influenced by the above reactions. The $n\nu$-interactions taking place within time interval Δt in a given commoving volume, containing N_n neutrons, reduce their total number by amount

$$\Delta N_n = -\left(\sum_{\epsilon_\nu} \sigma_{n\nu} n_{\epsilon\nu} v_\nu \Delta g_{\epsilon\nu} \right) N_n \Delta t, \tag{6}$$

---

[3] This correction accounts for the possibility that the gluons binding quarks inside the nucleon can split into quark–antiquark pairs, which could give nonvanishing contribution to the weak coupling.

[4] The Fermi constant can be determined with a very good accuracy measuring the life time of the muon, while g_A can be found only if one considers the interaction involving the nucleons.

where $n_{\epsilon_\nu} = (1 + \exp(-\epsilon_\nu/T_\nu))^{-1}$, $v_\nu = 1$ is the speed of the neutrinos, and Δg_{ϵ_ν} is the phase volume element:

$$\Delta g_{\epsilon_\nu} = \frac{1}{2\pi^2} \int_\nu^{\epsilon_\nu + \Delta\epsilon_\nu} |\mathbf{p}|^2 d|\mathbf{p}| \simeq \frac{1}{2\pi^2} \sqrt{(\epsilon_\nu^2 - m^2)} \epsilon_\nu \Delta\epsilon_\nu.$$

Introducing the relative concentration of the neutrons

$$X_n = \frac{N_n}{N_n + N_p} = \frac{n_n}{n_n + n_p}, \tag{7}$$

and substituting (5) in (6) we finally obtain the following expression for the rate of change of the neutron concentration due to $n\nu$-processes

$$\left(\frac{dX_n}{dt}\right)_{n\nu} = -\lambda_{n\nu} X_n = -\frac{1 + 3g_A^2}{2\pi^3} G_F^2 Q^5 J(1;\infty) X_n, \tag{8}$$

where

$$J(1;\infty) = \int_1^\infty dq q^2 (q-1)^2 \left(1 - \frac{(m_e/Q)^2}{q^2}\right)^{1/2} \left[1 + e^{\frac{Q}{T_\nu}(q-1)}\right]^{-1} \left[1 + e^{-\frac{Q}{T}q}\right]^{-1}, \tag{9}$$

and we have introduced the integration variable $q = (\epsilon_\nu/Q) + 1 = \epsilon_e/Q$. These expressions are given in Weinberg (1972). If we neglect the last multiplier into the integrand[5] and, take into account $q > 1$ and $(m_e/Q)^2 \simeq 0.15$, expand the square root keeping only first two terms, the obtained integrals can be exactly calculated and the result is

$$J(1;\infty) \simeq \frac{45\zeta(5)}{2} y^5 + \frac{7\pi^4}{60} y^4 + \frac{3\zeta(3)}{2} \left(1 - \frac{1}{2}\left(\frac{m_e}{Q}\right)^2\right) y^3, \tag{10}$$

where $y = T_\nu/Q$. It is quite remarkable that this approximate expression reproduces the exact result with very high accuracy at all temperatures. For instance, at $y > 1$ the accuracy is about 2%, improving to 1% and much better for $y < 1$.

It is not difficult to understand why it is the case. Actually at low temperatures ($y \ll 1$) this should be so since $\exp(-Q/T) \ll 1$.[6] On the other hand in the limit of very high temperatures ($y \gg 1$) the integral (9) can be very well approximated if one neglects (m_e/Q)- and (Q/T_ν)-terms; the result is

$$J(1;\infty) \simeq \frac{7\pi^4}{30} y^5 \quad \text{at} \quad y \gg 1. \tag{11}$$

Comparing this with the first term in (10), which obviously dominates in this limit, we see that they coincide within 3%-accuracy since $(45\zeta(5)/2 : (7\pi^4/30) = 1.027$. One can check numerically that in the intermediate range the accuracy of

---

[5] This means that one ignores the Pauli's exclusion principle.
[6] We remind that before $e^\pm$-annihilation $T = T_\nu$ and after that $T = 1.4T_\nu$.

the approximate expression (10) is better than 2%; for instance, at $y = 0.7$ it is about 1%.

Substituting (10) together with the numerical values of G_F, Q, expressed first in the Planck's units, into (8) and then returning back to the usual units ($[\lambda] = s^{-1}$) we infer that

$$\lambda_{n\nu} \simeq 1.63 y^3 (y + 0.25)^2 \, s^{-1}. \tag{12}$$

In this last expression further simplifications were made. However, the reader can check himself that at all temperatures, $T_\nu \geq 0.2$ MeV, its accuracy is never worse than 2–3%. Taking into account the experimental uncertainties in g_A this accuracy looks very satisfactory.

Similarly, we find that the rate of the reaction $n + e^+ \rightarrow p + \bar{\nu}$ is equal to

$$\lambda_{ne} = \frac{1 + 3g_A^2}{2\pi^3} G_F^2 Q^5 J\left(-\infty; -\frac{m_e}{Q}\right), \tag{13}$$

where J is the integral defined in (9) with the limits of integration from $-\infty$ to $-(m_e/Q)$. If $T_\nu = T$ and $T > m_e$, then $\lambda_{ne} \simeq \lambda_{n\nu}$.

The rates of the inverse reactions: $pe^- \rightarrow n\nu$ and $p\bar{\nu} \rightarrow ne^+$ are related to the rate of the direct reactions (at $T_\nu = T$) as

$$\lambda_{pe} = \exp(-Q/T)\lambda_{n\nu}, \quad \lambda_{p\nu} = \exp(-Q/T)\lambda_{ne}, \tag{14}$$

"**Freeze-out**": The inverse reactions lead to the increase of the neutron concentration with the rate $\lambda_{p\rightarrow n} X_p$; hence we can write the following balance equation for X_n:

$$\frac{dX_n}{dt} = -\lambda_{n\rightarrow p} X_n + \lambda_{p\rightarrow n} X_p = -\lambda_{n\rightarrow p}\left(1 + e^{-\frac{Q}{T}}\right)\left(X_n - X_n^{eq}\right), \tag{15}$$

where $\lambda_{n\rightarrow p} = \lambda_{ne} + \lambda_{n\nu}$ is the total rate of the direct reactions (1) and $X_n^{eq} = (1 + \exp(Q/T))^{-1}$. In deriving (15) I took into account that the proton concentration $X_p = 1 - X_n$ and used the relation (14) assuming that $T_\nu = T$.

The exact solution of this linear differential equation, with the initial condition $X_n \rightarrow X_n^{eq}$ as $t \rightarrow 0$, can be written in the following form

$$X_n(t) = X_n^{eq}(t) - \int_0^t d\tilde{t} \dot{X}_n^{eq} \, \exp\left(-\int_{\tilde{t}}^t \lambda_{n\rightarrow p}\left(1 + e^{-\frac{Q}{T}}\right) d\bar{t}\right), \tag{16}$$

where dot denotes the derivative with respect to time. At small t the second term in this equation, characterizing the deviations from the equilibrium, is negligible compared to the first one. Integrating by parts, one gets that in this limit the solution (16) can be rewritten as an asymptotic series in terms of increasing powers of the derivatives of X_n^{eq}

$$X_n = X_n^{eq}\left(1 - \frac{1}{\lambda_{n\rightarrow p}}\left(1 + e^{-\frac{Q}{T}}\right)^{-1} \frac{\dot{X}_n^{eq}}{X_n^{eq}} + \cdots\right). \tag{17}$$

Therefore, if $|\dot{X}_n^{\text{eq}}/X_n^{\text{eq}}| \sim t^{-1} \ll \lambda_{n \to p}$, that is the rate of the reactions is very high compared to the inverse cosmological time, $X_n = X_n^{\text{eq}}$, in complete agreement with the thermodynamical result. Much later, when the temperature significantly drops the "equilibrium concentration term" X_n^{eq} goes to zero and at the same time the integral on the right hand side of (16) approaches the finite limit. As a result the neutron concentration, instead of vanishing, as it would be in a state of local equilibrium, freeze-out at some value $X_n^* = X_n(t \to \infty)$.[7] The freeze-out effectively occurs when the second term in (17) is of the order of the first one, that is, when the deviations from the equilibrium become significant. Assuming that this happens before $e^{\pm}$-annihilation and after temperature drops below $Q \simeq 1.29$ MeV (these assumptions can be checked a posteriori) one can put $\lambda_{n \to p} \simeq 2\lambda_{n\nu}$ and neglect $\exp(-Q/T)$ in the obtained expressions. In this case the condition $|\dot{X}_n^{\text{eq}}/X_n^{\text{eq}}| \simeq \lambda_{n \to p}$, defining the freeze-out temperature T_*, takes the form

$$y_*^2 (y_* + 0.25)^2 \simeq 0.18 \kappa^{1/2}, \tag{18}$$

where $y_* = T_*/Q$. In deriving (18) I used the formula (12) for $\lambda_{n\nu}$ and took into account the relation between the temperature and cosmological time:

$$t_{\text{s}} = t_{\text{Pl}} \left(\frac{3}{32\pi\kappa}\right)^{1/2} \left(\frac{T_{\text{Pl}}}{T}\right)^2 \simeq 1.39 \kappa^{-1/2} \frac{1}{T_{\text{MeV}}^2}, \tag{19}$$

where $\kappa \equiv \frac{\pi^2}{30}(g_{\text{b}} + \frac{7}{8}g_{\text{f}})$ and g_{b}, g_{f} are the total number of the internal degrees of freedom, correspondingly, of all relativistic bosons and fermions.

In the case of three types of neutrino ($\kappa \simeq 3.54$) $y_* \simeq 0.65$ and the freeze-out temperature is $T_* \simeq 0.84$ MeV. The equilibrium neutron concentration at this moment is $X_n^{\text{eq}}(T_*) \simeq 0.18$. Of course, this number gives only a very rough idea about expected freeze-out concentration. One should not forget that at this moment the deviations from equilibrium are already very big and, in fact, $X_n(T_*)$ exceed the equilibrium concentration at least twice. The most important thing which could be learned from this simple estimate is that the freeze-out temperature depends on the number of light species present in the universe at this time. Since $T_* \propto \kappa^{1/8}$, the more light species are present, the bigger is the freeze-out temperature and one can expect that more neutrons will survive after chemical decoupling from the protons.[8] In turn, later on nearly all these neutrons build ^{4}He; hence one can expect that if, for instance, in addition to known types of neutrino, there exist the other light particles, then the abundance of the primordial helium should be higher than in the case of three neutrinos. This can be easily explained if we take into account that the rate of the expansion of the universe ($H = 1/2t$) at the given temperature increases

---

[7] Note that if λ would be decreasing not so fast, such that the integral in the exponent would diverge as $t \to \infty$, then the overall integral term would also vanish in this limit.

[8] If we use for freeze-out the simple criteria $t \simeq 1/\lambda$ then we get $T_* \propto \kappa^{1/6}$, the result which is usually quoted in the literature.

if we have extra light particles (see (19)); hence the freeze-out should occur earlier, when the neutron concentration is higher. For instance, in the extreme case of very big number of unknown light particles $T_* \gg Q$ and the expected concentration of the survived neutrons should be close to 50%, that is, there is one neutron per every proton. Later on these neutrons would bind the protons converting nearly all baryonic matter into ^{4}He. Of course, this would be in obvious conflict with the observational abundances of the light elements. Therefore, as we will see later, the primordial nucleosynthesis allows us to put rather strong restrictions on the number of light species.

Now I calculate the freeze-out concentration more accurately. Since $X_n^{\text{eq}} \to 0$ as $T \to 0$, this concentration is given by the integral term in (16) where we have to take the limit $t \to \infty$. Changing the integration variable from t to $y = T/Q$ (see (19)) and taking into account that the main contribution to the integral comes at $T > m_e$, where $\lambda_{n \to p} \simeq 2\lambda_{n\nu}$ and $\lambda_{n\nu}$ is given by (12), we obtain

$$X_n^* = \int_0^\infty \frac{dy}{2y^2(1 + \cosh(1/y))} \exp\left(-5.42\kappa^{-1/2} \int_0^y dx(x + 0.25)^2(1 + e^{-1/x})\right).$$
(20)

For the case of three neutrinos ($\kappa \simeq 3.54$) we obtain $X_n^* \simeq 0.158$. It is in very good agreement with the results of more elaborated numerical calculations. The presence of extra light neutrino increases κ by $2 \cdot \Delta\kappa_f \simeq 0.58$ and respectively the freeze-out concentration becomes $X_n^* \simeq 0.163$. Hence, two extra fermionic degrees of freedom (one for neutrino and one for antineutrino) lead to the increase of the freeze-out concentration by 0.5%.

Neutron decay: In the above consideration I have neglected the instability of the neutron via decay

$$n \to p + e^- + \bar{\nu}.$$
(21)

It was justified since the lifetime of free neutron $\tau_n = 885.7 \pm 0.8$ s is rather large compared to the typical cosmological time at the moment of freeze-out ($t_* \sim O(1)$ s). However, later on the two-body reactions (1) and inverse three-body reaction (21) become unimportant and the only remaining reaction reducing the amount of the neutrons is the neutron decay. As a result the relative concentration of the neutrons at $t \gg t_*$ is

$$X_n(t) = X_n^* \exp(-t/\tau_n).$$
(22)

Note that at late times one can neglect the degeneracy of the leptons which would increase the lifetime of the free neutrons; hence we can use the measured in the laboratory lifetime of the neutron quoted above. As we will see later the nucleosynthesis, as a result of which nearly all neutrons are captured in the nuclei, where they become stable, happens around $t \sim 200$ s. It is a rather substantial fraction of the neutron lifetime and therefore the neutron decay changes

significantly the amount of the survived neutrons and is important for the final ^{4}He-abundance.

3. DEUTERIUM BOTTLENECK

Complex nuclei are formed as a result of nuclear interactions of the baryons. For instance, ^{4}He could, in principle, be directly formed in many-body collisions; $2p + 2n \rightarrow {}^4$He. However, the number densities at the time when this reaction can take place are too low and its rate is negligible compared to the rate of expansion. Hence, the light complex nuclei can be produced only in sequence of two-body reactions. The first step on this way is the deuterium (D) production:

$$p + n \rightleftarrows D + \gamma. \tag{23}$$

There is no problem with this step since the rate of this reaction is very high and the "typical collision time" is, for sure, much smaller than the cosmological time (at $t < 10^3$ s). Hence one can expect that the deuterium should be in the local chemical equilibrium with nucleons. Let us define the deuterium abundance by weight as $X_D \equiv 2n_D/n_B$, where n_B is the total number of all baryons (nucleons) including those ones entering the complex nuclei. In the state of local chemical equilibrium the relation between X_D and the abundances of the free neutron and protons (appropriately, $X_n \equiv n_n/n_B$ and $X_p \equiv n_p/n_B$) can be easily found with the help of the equilibrium Saha's formula (see, for instance, Kolbe and Turner (1993):

$$X_D = 5.67 \times 10^{-14} \eta_{10} T_{\text{MeV}}^{3/2} \exp\left(\frac{B_D}{T}\right) X_p X_n, \tag{24}$$

where $B_D \equiv m_p + m_n - m_D \simeq 2.23$ MeV is the binding energy of the deuterium and the temperature is expressed in MeV. We have introduced here the normalized baryon-to-photon ratio

$$\eta_{10} \equiv \eta/10^{-10} = \left(\frac{n_B}{n_\gamma}\right)/10^{-10}, \tag{25}$$

which is related to the baryon contribution to the critical energy density Ω_B as

$$\Omega_B h_{75}^2 \simeq 6.53 \times 10^{-3} \eta_{10}, \tag{26}$$

where the Hubble constant h_{75} is normalized on 75 km/s·Mpc. The abundance of deuterium at the temperatures about its binding energy is still very small. For instance, for $T \sim 0.5$ MeV, we get $X_D \sim 2 \times 10^{-13}$. The reason for that is a very high entropy (number of photons) per baryon. Even at $T \ll B_D$ there are still enough highly energetic photons with $\epsilon > B_D$ which destroy the deuterium.

Actually the number of these photons per one nuclei of the deuterium is about

$$\frac{n_\gamma(\epsilon > B_{\mathrm{D}})}{n_{\mathrm{D}}} \sim \frac{B_{\mathrm{D}}^2 T e^{-B_{\mathrm{D}}/T}}{n_{\mathrm{B}} X_{\mathrm{D}}} \sim 10^{10} \frac{1}{\eta_{10} X_{\mathrm{D}}} \left(\frac{B_{\mathrm{D}}}{T}\right)^2 e^{-B_{\mathrm{D}}/T}. \qquad (27)$$

This number drops below unity at $T < 0.06$ MeV. Hence one can expect that the deuterium can be formed in significant amount only when the temperature is low enough, otherwise it is destroyed by the energetic photons. This also delays the formation of the other light elements as, for instance, ^{4}He.

The binding energy of the helium-4 (28.3 MeV) is much higher that the binding energy of the deuterium; hence if helium would be in the statistical equilibrium with neutrons and protons then one would expect that nearly all free neutrons would be captured in ^{4}He already at the temperature $\sim$0.3 MeV. However in reality the helium abundance is still negligible at this temperature. This is because the rates of the reactions converting deuterium in more heavy elements is proportional to the deuterium concentration and is much smaller than the expansion rate until the deuterium abundance will increase and constitute the substantial fraction of the baryonic matter. Before that only the protons, neutrons, and deuterium are in chemical equilibrium with each other. More heavy elements are decoupled and present in completely negligible amounts in spite of their high binding energies. This is known as "deuterium bottleneck." The size of the "bottleneck" which is proportional to X_{D} should become big enough to allow the neutrons and protons "to go through" and replenish the helium abundance in accordance with its high "equilibrium demand." Let us find when this happens. Using formula (24) we can express the temperature as a function of X_{D}:

$$T_{\mathrm{MeV}}(X_{\mathrm{D}}) \simeq \frac{0.061}{(1 + 2.7 \times 10^{-2} \ln(X_{\mathrm{D}}/\eta_{10}))}. \qquad (28)$$

This relation is valid only when the deuterium is in chemical equilibrium with neutrons and protons, which as we will see is true until the moment when X_{D} reaches the value 10^{-2}. According to the formula (28) the deuterium abundance should change from 10^{-5} to 1 when the temperature drops only in 1.5 times, namely, from 0.09 MeV to 0.06 MeV (for $\eta_{10} = 1$). Therefore, the deuterium abundance should increase very abruptly around this time and one can expect that the nuclear reactions should become fast enough to proceed with formation of light elements. The main processes converting the deuterium in more heavy elements are (see also Fig. 1):

$$1) \, \mathrm{D} + \mathrm{D} \rightarrow \, ^3\mathrm{He} + n, \quad 2) \, \mathrm{D} + \mathrm{D} \rightarrow T + p \qquad (29)$$
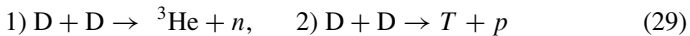
The cross-sections of these reactions are known from experiments and the results are usually presented in terms of the effective rates versus temperature.[9] In the

---

[9] The appropriate rates are cited in (Esmailzadeh *et al.* 1991). More recent data can be found on internet.
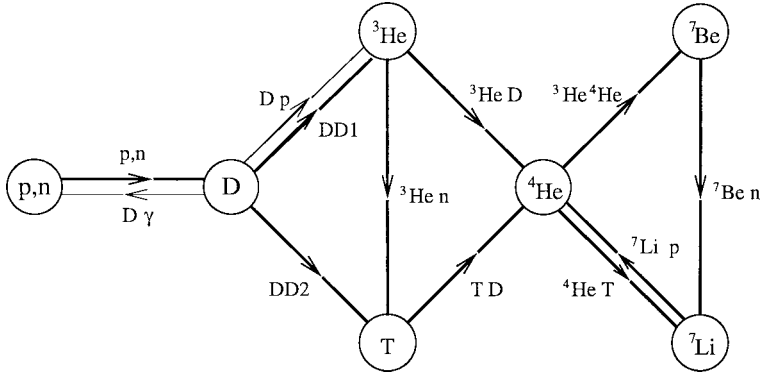
Fig. 1.

temperature interval $0.06 \div 0.09$ MeV these rates change not very much and we have

$$\langle \sigma v \rangle_{\text{DD1}} = (1.3 \div 2.2) \times 10^{-17} \, \text{cm}^3/\text{s}, \qquad \langle \sigma v \rangle_{\text{DD2}} = (1.2 \div 2) \times 10^{-17} \, \text{cm}^3/\text{s}.$$

$$(30)$$

Considering the comoving volume containing N_{D} deuterium nuclei we find that the decrease of their number during the time interval Δt due to the reaction (29) is equal to

$$\Delta N_{\text{D}} = -\langle \sigma v \rangle_{\text{DD}} n_{\text{D}} N_{\text{D}} \Delta t. \tag{31}$$

Rewriting this equation in terms of the concentration by weight $X_{\text{D}} \equiv 2N_{\text{D}}/N_{\text{B}}$ we obtain

$$\Delta X_{\text{D}} = -\frac{1}{2} \lambda_{\text{DD}} X_{\text{D}}^2 \Delta t, \tag{32}$$

where

$$\lambda_{\text{DD}} = (\langle \sigma v \rangle_{\text{DD1}} + (\langle \sigma v \rangle_{\text{DD2}}) n_{\text{B}} \sim 1.3 \times 10^5 K(T) T_{\text{MeV}}^3 \eta_{10} \, \text{s}^{-1} \tag{33}$$

and $K(T)$ is the numerical coefficient which changes from $\simeq 1$ to $\simeq 0.6$ when the temperature drops from 0.09 MeV to 0.06 MeV. It is clear that the substantial amount of the available deuterium can be converted into helium-3 and tritium within the cosmological time t only if $\Delta X_{\text{D}} \simeq (1/2)\lambda_{\text{DD}} X_{\text{D}}^2 t \sim X_{\text{D}}$; hence the "deuterium bottleneck opens wide" only when the deuterium concentration reaches the value

$$X_{\text{D}}^{(i)} \simeq \frac{1.2 \times 10^{-5}}{\eta_{10} T_{\text{MeV}}(X_{\text{D}})} \simeq 1.5 \times 10^{-4} \eta_{10}^{-1}(1 - 7 \times 10^{-2} \ln \eta_{10}). \tag{34}$$

Deriving this formula I used the relations (19), with $\kappa \simeq 1.11$, and (28); the obtained equation was solved by iterations assuming that $10^{-1} < \eta_{10} < 10$.

After deuterium abundance reaches the value given by (34) everything proceeds very fast. In fact, if $\eta_{10} = 1$ then according to (28) the equilibrium concentration X_D should increase from 10^{-4} to 10^{-2} when the temperature drops from 0.08 MeV to 0.07 MeV. This increase of X_D means that the reaction rates converting the deuterium to more heavy elements, which are proportional to X_D^2, at $T \sim 0.07$ MeV become 10^4 times bigger than the rate of the expansion. It is clear that this system is far from the equilibrium and the deuterium supplied by pn-reactions "is converted" very fast to more heavy elements. This doesn't allow the deuterium concentration to increase to the values bigger than 10^{-2}. The details of the nonequilibrium processes are described by a complicated system of kinetic equations which can be solved only numerically. In Fig. 2 the result of numerical calculations for the time evolution of the element abundances in the universe with $\Omega_B h_{75}^2 \simeq 5 \times 10^{-2}$ are shown (Burles *et al.*, 1999).

Below I present the calculations which explain the time behavior of these abundances and derive the formulae for the final freeze-out abundances of light elements up to ^{7}Be. This includes ^{4}He, deuterium (D), helium-3 (^{3}He), tritium (T), litium-7 (^{7}Li), and beryllium (^{7}Be). The other light elements as, for instance, ^{8}Li, ^{8}B et cetra are produced in very small amounts and will be ignored.

The most important nuclear reactions involving the light elements are schematically depicted in Fig. 1, which I recommend to keep in front of the eyes reading the rest of the paper.
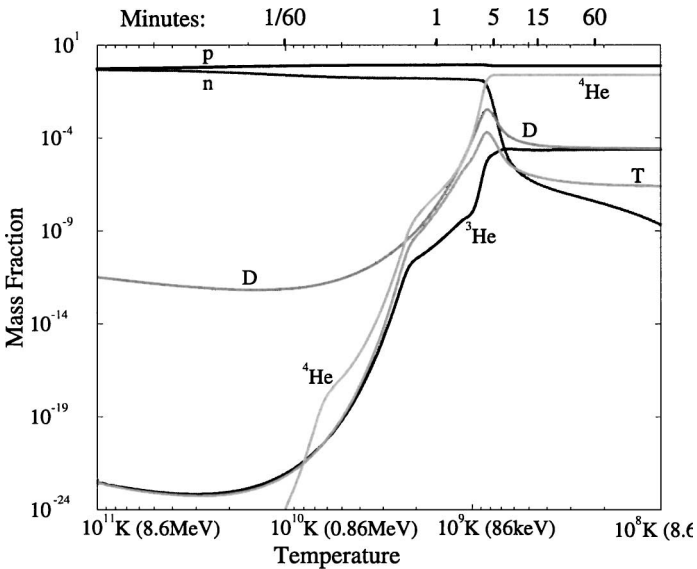

Fig. 2.

In Fig. 1 every element corresponds its own "reservoir." All these "reservoirs" are connected by "one-way-pipes." Every "pipe" corresponds to an appropriate nuclear reaction. I write only the initial elements involved in the reaction, since the outcome can be easily inferred from the picture. The "thickness of the pipe" through which the element a "escape from the reservoir" as a result of the reaction $ab \rightarrow cd$ is proportional to rate of this reaction

$$\dot{X}_a / X_a = -A_b^{-1} \lambda_{ab} X_b, \tag{35}$$

where $\lambda_{ab} = \langle \sigma v \rangle_{ab} n_B$ and A_b is the mass number of the element b; for instance, $A = 4$ for ^{4}He and $A = 7$ for ^{7}Li, ^{7}Be. Of course, the appropriate "pipe" is efficient only if $\dot{X}_a / X_a > t^{-1}$. As we have already seen the "D- and p, n-reservoirs" are in equilibrium with each other and decoupled from the rest at the temperatures above 0.08 MeV (deuterium bottleneck). However when the temperature drops below 0.08 MeV the "DD-pipes open" and become very efficient in converting an extra deuterium supply from "np-reservoir" into more heavy element. Finally nearly all free neutrons disappear entering more heavy elements where they become stable. After that the concentrations of the elements in the appropriate "reservoirs" freeze-out and the "final abundances" survive. This is a general picture and now I proceed with detailed calculations and consider the formation of every element separately.

4. HELIUM-4

As soon as deuterium concentration increases to $X_D^{(i)}$ given by (34) the formation of the other light elements begins. This happens at the temperature (see (28))

$$T_{MeV}^{(i)} \simeq 0.08(1 + 7 \times 10^{-2} \ln \eta_{10}), \tag{36}$$

at the moment of time[10]

$$t_s^{(i)} \simeq 206 \left(\frac{\kappa}{1.11} \right)^{-1/2} (1 - 0.14 \ln \eta_{10}). \tag{37}$$

Of course, the nucleosynthesis does not happen instantaneously. Moreover at the beginning the rate of deuterium production in reaction, $pn \rightarrow D\gamma$, is substantially higher that the total rate of the deuterium "annihilation" in reaction (29), namely,

$$\frac{\lambda_{pn} X_p X_n}{\lambda_{DD} X_D^2} \simeq 10^4 \left(\frac{10^{-4}}{X_D} \right)^2, \tag{38}$$

where I used the experimental value for the ratio $\lambda_{pn}/\lambda_{DD}$, which is about 10^{-3} at $T_{MeV} \simeq 0.07 \div 0.08$ and put $X_n = 1 - X_p \simeq 0.16$.

---

[10] Note that $T^{(i)}$ and $t^{(i)}$ depend on the exact value of $X_D^{(i)}$ only logarithmically and therefore not very sensitive to the exact value of $X_D^{(i)}$.

As it follows from (38) before the deuterium concentration reaches its maximal value $X_D \sim 10^{-2}$ the deuterium production dominates over deuterium destruction and the deuterium abundance continues to follow its chemical equilibrium track given by (24). According to (28) the concentration $X_D \simeq 10^{-2}$ is reached very fast after $t^{(i)}$, namely, when the temperature drops from 0.08 MeV to 0.07 MeV (for $\eta_{10} = 1$), that is, with

$$\Delta t \simeq 2t^{(i)} \frac{\Delta T}{T^{(i)}} \simeq 50 \, \text{s} \tag{39}$$

time delay after $t^{(i)}$. When this concentration is reached the two-body DD-deuterium destruction become more efficient than the pn-deuterium production and X_D begins to decrease[11] (see Fig. 2).

The concentration of the free neutrons during this period strongly decreases and they go first to the "deuterium reservoir" and then proceed further "through the pipes" forming heavy elements. For most neutrons the "final destination" is the "^{4}He-reservoir."

Why it is so can be understood even without analyzing the rates of the intermediate reactions. Actually, if ^{4}He would be in the equilibrium with the other light elements it would be dominating at low temperatures because of its high binding energy (28.3 MeV), which is four times bigger than the binding energies of the intermediate elements, ^{3}He (7.72 MeV) and T (6.92 MeV). The system which is away from equilibrium always tends there in a quickest possible way. Therefore, most of the free neutrons will be captured into ^{4}He-nuclei because its equilibrium demand is the highest.

The reactions proceed in the following way. First, the deuterium is converted into ^{3}He and T in reaction (29). After that tritium interacts with deuterium and produce the helium-4 nuclei:

$$T + D \rightarrow {}^4He + n \tag{40}$$

As a result two neutrons out of three are captured into the ^{4}He-nuclei and one comes back into "np-reservoir."

The ^{3}He-nuclei can interact either with free neutrons and then proceed to "T-reservoir,"

$$^3He + n \rightarrow T + p, \tag{41}$$

or with deuterium going directly to "^{4}He-reservoir"

$$^3He + D \rightarrow {}^4He + p. \tag{42}$$

---

[11] The deuterium photo-destruction can be completely neglected after that. It is clear if we note that if there would be only photo-desctuction processes alone then the deuterium concentration would continue to increase.

The ratio of rates for these reactions is

$$\frac{\lambda_{^3\text{He}n} X_{^3\text{He}} X_n}{\lambda_{^3\text{He}D} X_{^3\text{He}} X_D} \sim 6 \frac{X_n}{X_D}; \tag{43}$$

hence at the beginning "^{3}HeD-pipe" is inefficient compared to "^{3}Hen-pipe" and most of ^{3}He-nuclei are converted to tritium. Only when the concentration of the free neutrons drops below the deuterium concentration (which is always smaller that 10^{-2}), the rate of the reaction (42) converting ^{3}He directly to ^{4}He becomes bigger that the rate of the reaction (41). It follows from here that most of the neutrons will go into ^{4}He-nuclei either along $np \rightarrow D \rightarrow T \rightarrow {}^4\text{He}$ or $np \rightarrow D \rightarrow {}^3\text{He} \rightarrow T \rightarrow {}^4\text{He}$ way. Finally, in about $50 \div 100$ s after the beginning of nucleosynthesis nearly all neutrons (with the exception of very small fraction $< 10^{-3}$), end up in ^{4}He-nuclei. Therefore, the final ^{4}He-abundance is completely determined by the amount of the available free neutrons at the time when DD-reactions become efficient, that is at $t \simeq t^{(i)}$. Because half of the total weight of ^{4}He is due to the protons, its final abundance by weight should be

$$X_{^4\text{He}}^f = 2X_n\big(t^{(i)}\big) = 2X_n^* \exp\left(-\frac{t^{(i)}}{\tau_n}\right). \tag{44}$$

Substituting here X_n^* from (20) and $t^{(i)}$ from (37) we obtain:

$$X_{^4\text{He}}^f = 2(0.158 + 0.005(N_v - 3)) \cdot \exp\left(-\frac{206}{886} \frac{(1 - 0.14 \ln \eta_{10})}{1 + \frac{0.15}{1.11}(N_v - 3))^{1/2}}\right)$$

$$\simeq 0.25 + 0.012(N_v - 3) + 0.0082 \ln \eta_{10}, \tag{45}$$

where N_v is the number of massless neutrino species. This result is in very good agreement with the results of the numerical calculations presented in Fig. 3 (Olive *et al.*, 2000).

   In fact, this agreement can be made even better if one notes that the formation of ^{4}He is not an instantaneous event which happens at $t^{(i)}$. It starts at $t^{(i)}$ and then continues for, at least, 50 s (see (39)). Most of the neutrons are trapped at the end. Therefore the time delay reduces the amount of ^{4}He to $X_{^4\text{He}} \simeq 0.25 \exp(-50/886) \simeq 0.236$ that is by 1.4%.

   As we see from (45) the abundance of ^{4}He depends on the number of massless species N_v. The presence of extra massless neutrino increases the ^{4}He-abundance by 1.2%. There are two reasons for this. Two-third out of this increase is due to the dependence of the freeze-out concentration X_n^* from the number of the massless species. In fact, more species one has, more fast universe expands at given temperature and hence the freeze-out of the neutrons occurs earlier, when their concentration is higher. The remaining one-third has a similar nature. Namely, for given baryon density the nucleosynthesis happens at appropriate temperature. This temperature is reached earlier if there are more light species and therefore
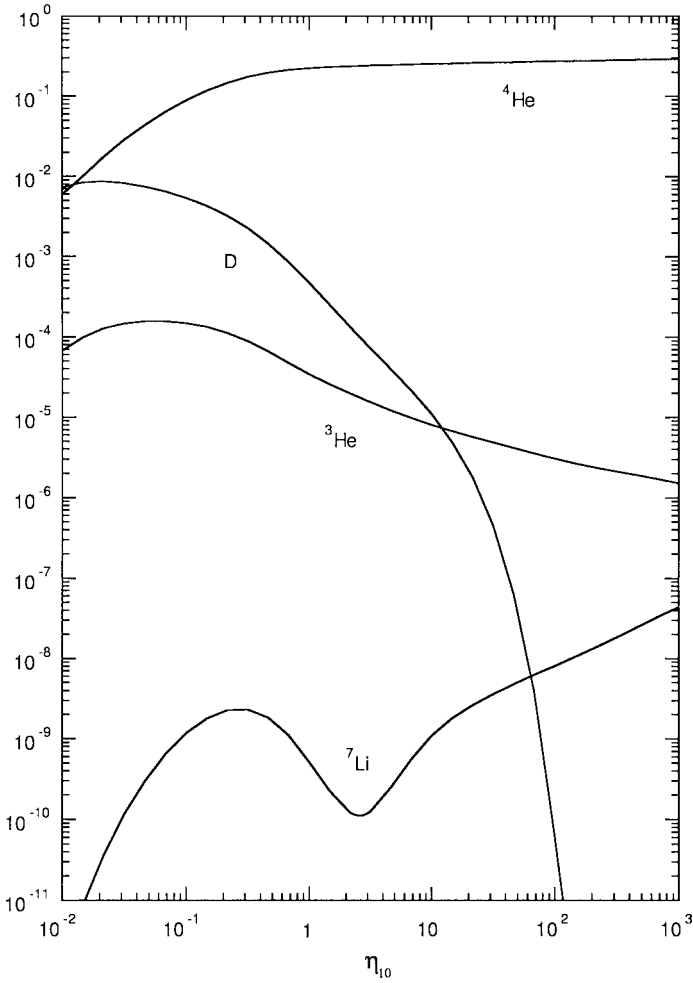
Fig. 3.

more neutrons survive until they will be captured. The dependence of the ^{4}He-abundance on the number of light species taken together with the results for the deuterium abundance allows us to put rather strong bounds on the number of unknown light particles which were relativistic at the time of nucleosynthesis.

The helium abundance also depends on the baryon density (entropy per baryon) and according to (45) increases by $\sim$2% (numerical result $\simeq$2.5%) if the density is 10 times higher. The physical origin of this dependence is very clear. In the universe with bigger concentration of baryons the nucleosynthesis begins

earlier, at higher temperature (see (36)); hence more neutrons survive till this time and more Helium-4 is formed.

5. DEUTERIUM

To calculate the time evolution and freeze-out concentration of the deuterium I will make some assumption which significantly simplify the consideration. The validity of these assumption can be checked a posteriori.

First of all, I ignore ^{7}Be, ^{7}Li since their abundances as we will see later are always small compared to the abundances of ^{3}He and T. Second, I will assume that ^{3}He and T abundances always have quasi-equilibrium values, which are determined by condition that the "total incoming in appropriate reservoir flux should be equal to the outgoing flux"[12] (see Fig. 1). For instance, in the case of ^{3}He it means that the amount of ^{3}He produced within some time interval in DD and Dp-reactions should be equal to the amount of ^{3}He destroyed within the same time in ^{3}HeD and ^{3}Hen-reactions. This is well justified because the rate of the reactions in which ^{3}He is destroyed is high enough to take care about "quick adjustment" of ^{3}He-concentration to the change of deuterium abundance.

The system of reservoirs with pipes, depicted in Fig. 1 is a "self-regulated system" with small adjustment time. The overall picture after the beginning of the nucleosynthesis is the following. When deuterium concentration reaches $X_D \simeq 10^{-2}$ the rate of DD-reactions become comparable with the rate of the deuterium production via pn-interactions (see (43)) and then dominates. The neutrons are taken from "np-reservoir" and send via "D-reservoir" along "DD- and Dp-pipes" first to "^{3}He and T-reservoirs" and from there through "^{3}HeD and TD-pipes" to their final destination, namely, in "^{4}He-reservoir." Not all of the neutrons taken from "np-reservoir" reach the "^{4}He-reservoir" in the first try. Some of them "escape" on the way there. Namely, in "DD1 and TD-pipes" one neutron is released in the reactions DD $\rightarrow$ ^{3}Hen, TD $\rightarrow$ ^{4}Hen, comes back to "np-reservoir" and then participate in the next try to get "^{4}He-reservoir." Thus after the beginning of nucleosynthesis there is a stationary flux of the neutrons from "pn-reservoir" to "^{4}He-reservoir" through the system of "pipes" via intermediate "D, ^{3}He, and T-reservoirs." The "widths of the pipes" (reaction rates) connected to ^{3}He and T-reservoirs depend on the concentration in the appropriate reservoir. For instance, the width of the "^{3}HeD and TD-pipes" is proportional, respectively, to ^{3}He and T-concentrations. If the amount of ^{3}He would increase/decrease compared to its quasi-equilibrium value the size of "^{3}HeD-pipe" would be quickly adjusted (respectively increases/decreases) to bring its concentration to the quasi-equilibrium value, which is determined by the condition of *zero total flux*.

If the universe would not be expanding then finally all neutrons would go to "^{4}He-reservoir" and as long as the temperature goes to zero, noting would be left

---

[12] This condition reminds the first Kirchhoff's rule for the electric currents.

besides the protons and ^{4}He. However, the expansion plays the role of "water-tap" for the "pipes." At the moment when the reaction rates become smaller than the rate of the expansion the "water-taps" close and the abundances of the elements in the appropriate reservoirs freeze-out at their quasi-equilibrium values. The final abundances of ^{3}He and T are determined by deuterium freeze-out concentration which we have to calculate.

Analyzing the system of kinetic equations one can find that even if ^{3}He and T have quasi-equilibrium concentrations the neutrons and deuterium concentration not necessary satisfy the quasi-equilibrium conditions. Therefore, we have to derive the equations which describe the time dependence of the appropriate abundances X_n, X_D after X_D reached the value $\sim 10^{-2}$.

The reaction rate for the elements a, b which is equal to $\lambda_{ab} n_a n_b / n_B^2$ can be rewritten in terms of the concentrations by weight as

$$\frac{1}{A_a A_b} \lambda_{ab} X_a X_b \tag{46}$$

where A_a, A_b are the mass numbers of the elements a and b. The quasi-equilibrium condition for ^{3}He takes then the following form:

$$\frac{1}{4}\lambda_{DD1} X_D^2 + \frac{1}{2}\lambda_{Dp} X_D X_p = \frac{1}{6}\lambda_{^3HeD} X_{^3He} X_D + \frac{1}{3}\lambda_{^3Hen} X_{^3He} X_n. \tag{47}$$

Similar for tritium we have

$$\frac{1}{4}\lambda_{DD2} X_D^2 + \frac{1}{3}\lambda_{^3Hen} X_{^3He} X_n = \frac{1}{6}\lambda_{TD} X_T X_D. \tag{48}$$

I will assume that these conditions are always satisfied.

The general kinetic equation for the rate of change of free neutrons concentration can be easily written if we take into account they are produced in the reactions DD $\rightarrow$ ^{3}Hen and DT $\rightarrow$ ^{4}Hen and "destroyed" in the processes $pn \rightarrow$ Dγ and ^{3}He$n \rightarrow$ Tp:

$$\frac{dX_n}{dt} = \frac{1}{4}\lambda_{DD1} X_D^2 + \frac{1}{6}\lambda_{TD} X_T X_D - \lambda_{pn} X_p X_n - \frac{1}{3}\lambda_{^3Hen} X_{^3He} X_n. \tag{49}$$

Assuming that tritium satisfies quasi-equilibrium condition (48) one can simplify this equation to

$$\frac{dX_n}{dt} = \frac{1}{4}\lambda_{DD} X_D^2 - \lambda_{pn} X_p X_n, \tag{50}$$

where as usual $\lambda_{DD} = \lambda_{DD1} = \lambda_{DD2}$.

The appropriate equation for the deuterium is similarly derived by using (47) and (48):

$$\frac{dX_D}{dt} = 2\lambda_{pn} X_p X_n - \lambda_{DD} X_D^2 - 2\lambda_{Dp} X_D X_p. \tag{51}$$

Expressing time through the temperature via (19) and substituting the numerical values for λ_{DD} given by (33), the above equations reduce to

$$\frac{dX_n}{dT_{\mathrm{MeV}}} = a \cdot K(T)\eta_{10}\big(R_1 X_n - X_{\mathrm{D}}^2\big) \tag{52}$$

and

$$\frac{dX_{\mathrm{D}}}{dT_{\mathrm{MeV}}} = 4a \cdot K(T)\eta_{10}\left(X_{\mathrm{D}}^2 + R_2 X_{\mathrm{D}} - \frac{1}{2}R_1 X_n\right), \tag{53}$$

where $a = 0.86 \times 10^5$ and the coefficient $K(T)$ accounts for the temperature dependence of $\langle \sigma v \rangle$ for DD-reactions and changes from ~ 1 to 0.5 when the temperature drops from 0.09 to 0.04 MeV. In the expressions

$$R_1 \equiv 4X_p\frac{\lambda_{pn}}{\lambda_{\mathrm{DD}}} \simeq (3 \div 8) \times 10^{-3}, \quad R_2 \equiv 2X_p\frac{\lambda_{p\mathrm{D}}}{\lambda_{\mathrm{DD}}} \simeq (2.5 \div 2.3) \times 10^{-5} \tag{54}$$

I used the experimental value for the ratio of the appropriate reaction rates; the first number within the brackets corresponds to the higher temperature when it changes in the interval $T_{\mathrm{MeV}} \simeq 0.09 \div 0.04$.

The system of equations (52) and (53) has an attractor solution, which can be easily found if we consider X_{D} as a function X_n (or vise versa) and rewrite the eqs. (52), (53) as

$$\frac{dX_{\mathrm{D}}}{dX_n} = 4\frac{\big(X_{\mathrm{D}}^2 + R_2 X_{\mathrm{D}} - \frac{1}{2}R_1 X_n\big)}{\big(R_1 X_n - X_{\mathrm{D}}^2\big)}.$$

If $X_{\mathrm{D}} \ll X_n$ then

$$X_n = \frac{2}{R_1}\big(X_{\mathrm{D}}^2 + R_2 X_{\mathrm{D}}\big)\left[1 - \frac{1}{8}\frac{X_{\mathrm{D}}}{X_n} + O\left(\left(\frac{X_{\mathrm{D}}}{X_n}\right)^2\right)\right] \tag{55}$$

satisfies this equation up to the second order terms in X_{D}/X_n. The solution (55) is a good approximate solution after deuterium concentration reaches the maximal value of about 10^{-2} and starts to decrease (see Fig. 2). It is valid until the moment when the neutron concentration drops to and becomes comparable to the deuterium concentration. The solution (55) describes the situation when the deuterium abundance satisfy the quasi-equilibrium condition. One can check that in this case the time derivative of the deuterium concentration in the l.h.s. of the equation (53) is small compared to every separate term in the r.h.s. of this equation. Since $R_2 \ll R_1$ we infer from the eq. (55) that the deuterium and neutron concentration become comparable when the deuterium concentration drops to $O(1)R_1$. Before this happens (for $X_{\mathrm{D}}, X_n > O(1)R_1$) the deuterium concentration can be expressed through the neutron concentration as

$$X_{\mathrm{D}} \simeq \sqrt{\frac{R_1 X_n}{2}} \tag{56}$$

Note that according to this formula the maximal possible concentration which deuterium can reach is $X_D \simeq 10^{-2}$ when most the free neutrons are still not captured by light elements ($X_n \simeq 0.12$). This is in complete agreement with naive estimate we got before comparing of pn and DD-reactions rates. When deuterium follows its quasi-equilibrium track (56) the neutrons concentration satisfies the equation

$$\frac{dX_n}{dT_{\text{MeV}}} \simeq \frac{1}{2} a \cdot K(T) \eta_{10} R_1 X_n. \tag{57}$$

In this case the neutrons are the "key element" which determines the quasi-equilibrium concentrations of all other elements including deuterium. In other words, the neutrons regulate the "water-taps in the pipes connecting the reservoirs in Fig. 1." The equation (57) starts to be applicable at the moment when deuterium concentration grows to 10^{-2}. At this time most of free neutrons are not yet trapped by the light elements and $X_n \simeq 0.12$. According to (28), which is still applicable at this time, the deuterium reaches the maximal possible 10^{-2} when the temperature drops to

$$T_{\text{MeV}} \simeq 0.07 + 0.002 \ln \eta_{10}. \tag{58}$$

After that the neutron concentration satisfies the equation (57), the approximate solution of which is

$$X_n(T_{\text{MeV}}) \simeq 0.12 \exp\left(\frac{1}{2} a \cdot K(T) \eta_{10} R_1 (T_{\text{MeV}} - 0.07 - 0.002 \ln \eta_{10})\right). \tag{59}$$

As it follows from here, the neutron concentration decreases as the temperature drops and becomes equal to the deuterium concentration ($\sim R_1$) when

$$T_{\text{MeV}}^* \sim 0.07 - 0.02 K_{T^*}^{-1} \eta_{10}^{-1} + 0.002 \ln \eta_{10} \tag{60}$$

It is clear that this formula is not applicable if $\eta_{10} < 0.35$. In the universe with very low baryon density ($\eta_{10} \ll 1$) the neutron concentration never drops below the deuterium concentration. It freezes-out before. In this case the nucleosynthesis is over very fast after beginning and neutron concentration freezes-out before the substantial part of the neutrons is converted into ^{4}He. After that the free neutrons decay. This explains why in the universe with very low baryon density (for instance, with $\eta_{10} \simeq 10^{-2}$) the helium abundance is less than 1% (see Fig. 3). When I was deriving the formula (45) for ^{4}He-abundance I assumed that the reactions converting the neutrons into ^{4}He are very efficient and able to transfer most of the available neutrons into more heavy elements. This means that this formula is valid only for $\eta_{10} > 0.35$. From the observations of the luminous baryonic matter we know that $1 < \eta_{10} < 10^2$ and therefore we concentrate from now on only on this range of parameter for η_{10}. All the derivations below will be done under this assumption.

The neutron concentration drops to $X_n \sim X_D \sim R_1$ at the temperature T^* given by (60). After that the solution (56) is not valid anymore and the system quickly gets to another attractor which correspond to the quasi-equilibrium solution of the equation (52), namely,

$$X_n = \frac{1}{R_1} X_D^2 \left[1 + 4\frac{X_n}{X_D} + O\left(\left(\frac{X_n}{X_D}\right)^2\right) \right] \tag{61}$$

As long as X_n/X_D continues to drop one can neglect the deviations from attractor solution $X_n \simeq X_D^2/R_1$ and the equation (53) for deuterium takes the following form

$$\frac{dX_D}{dT_{MeV}} = 2a \cdot K(T)\eta_{10}\left(X_D^2 + 2R_2 X_D\right) \tag{62}$$

Now the deuterium becomes the "key element" and determines its own "fate" regulating simultaneously the quasi-equilibrium concentrations of the other elements including the neutrons. Since R_2 practically doesn't change in the relevant temperature interval (see (54)) it can be treated as a constant and the equation (62) can be easily integrated:

$$\left(1 + \frac{2R_2}{X_D(T)}\right) = \left(1 + \frac{2R_2}{X_D(T^*)}\right) \exp\left(4a\eta_{10}R_2 \int_T^{T^*} K(T)dT\right), \tag{63}$$

where the temperature is expressed in MeV. When temperature goes to zero ($T \to 0$) the deuterium concentration doesn't vanish, instead it freezes-out at $X_D^f \equiv X_D(T \to 0)$. Taking into account that $X_D(T^*) \sim R_1 \gg R_2$ and estimating the integral in (63) as $\sim K(T^*)T^*$, where T^* is given by (60), we obtain the following expression for the deuterium freeze-out concentration:

$$X_D^f \simeq \frac{2R_2}{(\exp(A\eta_{10}) - 1)}, \tag{64}$$

where

$$A \equiv 2aR_2 K(T^*)T^*. \tag{65}$$

The numerical coefficient A only slightly varies with η_{10}. Actually when η_{10} changes by two decade from 1 to 10^2 this coefficient increases only twice from ~ 0.1 to ~ 0.2. The expression (64) fits very well with the results of the numerical calculations presented in Fig. 3 If we want to get better accuracy using analytical approach we can do it taking into account the temperature dependence of the reaction rates. The formula (64) is in satisfactory agreement with the numerical results if we take $A = 0.1 = $ const. At $\eta_{10} < 1/A \sim 10$ the good approximation for (64) is

$$X_D^f \simeq \frac{2R_2}{A\eta_{10}} \sim 4 \times 10^{-4}\eta_{10}^{-1}. \tag{66}$$

We see that in this range of η_{10} the deuterium freeze-out concentration decreases nearly linearly with η_{10}. It is easy to understand. In this case the freeze-out concentration never drops below $R_2 \sim 10^{-5}$ and as it is clear from the equation (62) DD-reactions always dominate over Dp-reactions in destroying deuterium. Therefore, the deuterium concentration freeze out when $\lambda_{DD}X_D^f \sim t^{-1}$; since $\lambda_{DD} \propto n_B \propto \eta_{10}$ we see that in the leading order X_D^f should be inversely proportional to η_{10} (compare to (34)).

On the contrary, if $\eta_{10} > 10$ the linear in X_D-term in the equation (62) dominates after X_D drops below $R_2 \sim 10^{-5}$ and after that $X_D \propto \exp(-\eta_{10} \times$ function of T); hence as it follows from (64) the freeze-out concentration in this limit is

$$X_D^f \simeq 2R_2 \exp(-A\eta_{10}) \tag{67}$$

and decays by five order of magnitude from $\sim 10^{-5}$ to $\sim 10^{-10}$ when η_{10} changes by only one decade from 10 to 100 (see Fig. 3). In this case the freeze-out concentration is entirely determined by the reaction $Dp \to {}^3\text{He}\gamma$ which dominates over DD-deuterium destruction during the last stage before freeze-out. Hence in a dense universe nearly all deuterium is very efficiently burned down in this reaction. The deuterium abundance is very sensitive indicator of the baryon density. This allows us to put rather strong upper bond on η_{10} from observations.

6. HELIUM-3 AND TRITIUM

Now we can calculate the freeze-out abundances of the other light elements using the quasi-equilibrium conditions and assuming that these conditions are still satisfied at the moment of deuterium freeze-out. First I consider ${}^3\text{He}$ and assume that $X_D^f > R_2 \sim 10^{-5}$, that is, I consider the case of $1 < \eta_{10} < 10$. In this case the deuterium freeze-out is determined by DD-reaction and happens at the time determined by condition $\lambda_{DD}X_D^f \sim t_D^{-1}$. The freeze-out time for ${}^3\text{He}$ can be estimated, requiring that the reaction ${}^3\text{HeD} \to {}^4\text{He}n$ becomes inefficient in converting the significant amount of ${}^3\text{He}$ into ${}^4\text{He}$. This occurs when $\lambda_{{}^3\text{HeD}}X_D \sim t_{{}^3\text{He}}^{-1}$. Since $\lambda_{{}^3\text{HeD}}$ is few times bigger than λ_{DD} it is clear that ${}^3\text{He}$ concentration freezes-out a little bit later than the deuterium concentration. This means that at the moment of deuterium freeze-out the ${}^3\text{HeD}$-reaction is still efficient in returning neutrons back to "np-reservoir"[13]; hence the quasi-equilibrium solution for the neutrons (61) derived under this assumption is still valid. Substituting $X_n = X_D^2/R_1$ in (47) we can express the quasi-equilibrium concentration of ${}^3\text{He}$ through X_D:

$$X_{{}^3\text{He}} = \frac{3}{2} \frac{\lambda_{DD1}X_D + 2\lambda_{Dp}X_p}{\lambda_{{}^3\text{HeD}} + 2(\lambda_{{}^3\text{He}n}/R_1)X_D}. \tag{68}$$

---

[13] The same is true for the tritium since $\lambda_{TD} \simeq \lambda_{\text{He}_3D}$. Also note that DD-reaction at this time still continue to actively participate in refilling the "np-reservoir" in accordance with small quasi-equilibrium neutron demand.

After deuterium freezes-out the small leakage from "D to ^{3}He-reservoir" is still able to keep stationary quasi-equilibrium "flow through ^{3}He-reservoir." Actually in the considered case the ^{3}He-concentration is significantly smaller than X_D^f and the deuterium demand needed to compensate the leakage of ^{3}He through "^{3}HeD-pipe" is not very high. After a short time when the ^{3}HeD-reaction becomes ineffcient both "DD1 and ^{3}HeD-pipes" close up simultaneously and the ^{3}He-freeze-out concentration can be obtained from (68) substituting there X_D^f given by (64)

$$X_{^3He}^f \sim \frac{0.2X_D^f + 10^{-5}}{1 + 4 \times 10^3 X_D^f}. \tag{69}$$

Here I first normalized all reaction rates on $\lambda_{^3HeD}$ and then used the experimental values for the obtained ratios. Of course these ratios depend on the temperature. However, as one can check, they do not change too much, namely, not more than by factor two in the whole range of the relevant temperatures for the two decade of baryon density. For the definiteness I took them at $T \sim 0.06$ MeV. The obtained result is in excellent agreement with the results of the numerical calculations presented in Fig. 3 We see that if $X_D^f \simeq 10^{-3}$ then the appropriate ^{3}He abundance is 10 times smaller that the deuterium abundance. It is clear from the above expressions that this suppression of ^{3}He-abundance compared to the deuterium abundance is mostly due to the reactions converting ^{3}He into tritium, which are still very efficient at the moment of freeze-out because of the rather high concentration of free neutrons. In more dense universe where the deuterium abundance is smaller the availability of the free neutrons appropriately reduces. If, for instance, X_D^f is smaller than $\sim 2.5 \times 10^{-4}$ (see the denominator in the formula (68) and (69)) then the reaction ^{3}He$n \to$ Tp does not play any significant role in determining the final ^{3}He-abundance. This is why we can still use (69) to estimate $X_{^3He}^f$ even in the universe with a relatively high baryon density, where $X_D^f < 10^{-5}$, although the free neutron concentration can significantly deviate from that given by (61).

From (69) it follows that, if $X_D^f \simeq 1.2 \times 10^{-5}$, then the helium-3 freeze-out concentration is equal to the deuterium concentration. This is in very good agreement with the numerical results. In the universe with $\eta_{10} > 10$ the helium-3 is produced in the reaction D$p \to$ ^{3}Heγ and destroyed in the process ^{3}HeD $\to$ ^{4}Hen. Irrespective of how big is X_D these two competing processes give rise to the final ^{3}He-abundance, $X_{^3He}^f \simeq D_p/\lambda_{^3HeD} \simeq 10^{-5}$ even in the case when the deuterium is practically absent. Slight deviation of the numerical results from predicted here constant ^{3}He-concentration in this limit is due to weak temperature dependence of the reaction rates.

Similarly, one can find that the tritium quasi-equilibrium concentration is equal to

$$X_T = \left(\frac{3}{2} \frac{\lambda_{DD2}}{\lambda_{DT}} + 2 \frac{\lambda_{^3Hen}}{\lambda_{DT}} \frac{X_{^3He}}{R_1} \right) X_D. \tag{70}$$

Using the experimental values for the ratio of the reaction rates $\lambda_{DD2}/\lambda_{DT} \simeq 0.01$ and $\lambda_{^3He n}/\lambda_{DT} \simeq 1$ we can easily understand the dependence of tritium freeze-out concentration on η_{10}. The formulae (68) and (70) explain why the ^{3}He-track in Fig. 2, in distinction from T, doesn't "repeat" the track of the deuterium, namely, the ^{3}He-concentration increases monotonically all the time, while T-concentration first reaches the maximum and then decreases until it gets its freeze-out value.

7. LITIUM-7 AND BERILIUM-7

The quasi-equilibrium concentrations of ^{7}Li and ^{7}Be can be determined from the equations

$$\frac{1}{12}\lambda_{^4HeT}X_{^4He}X_T + \frac{1}{7}\lambda_{^7Ben}X_{^7Be}X_n = \frac{1}{7}\lambda_{^7Lip}X_{^7Li}X_p \tag{71}$$

and

$$\frac{1}{12}\lambda_{^4He^3He}X_{^4He}X_{^3He} = \frac{1}{7}\lambda_{^7Ben}X_{^7Be}X_n, \tag{72}$$

where I took into account only dominating reactions in which, respectively, ^{7}Li and ^{7}Be are produced and destroyed. One can check that the other reactions, for instance, ^{7}LiD $\rightarrow n + 2^4$He and ^{7}BeD $\rightarrow p + 2^4$He for $\eta_{10} > 1$ can be ignored. It immediately follows from these equations that

$$X_{^7Li} = \frac{7}{12}\frac{X_{^4He}}{X_p}\left(\frac{\lambda_{^4HeT}}{\lambda_{^7Lip}}\right)\left(X_T + \frac{\lambda_{^4He^3He}}{\lambda_{^4HeT}}X_{^3He}\right). \tag{73}$$

The ratio of $\lambda_{^4HeT}/\lambda_{^7Lip}$ is remarkably constant within rather broad temperature interval, namely, it increases from $\sim 2.2 \times 10^{-3}$ to $\sim 3 \times 10^{-3}$ when the temperature drops three times from 0.09 to 0.03 MeV. In distinction from that $K_1(T) \equiv \lambda_{^4He^3He}/\lambda_{^4HeT}$ varies quite significantly in the same temperature interval: $K_1(T) \simeq (5 \div 0.6) \times 10^{-2}$, that is, it drops nearly 10 times when the temperature drops in three times. Substituting $X_{^4He} \simeq 0.25$, $X_p \simeq 0.75$ in (73) we obtain

$$X_{^7Li} \simeq (3 \div 5) \times 10^{-4}(X_T + K_1(T)X_{^3He}). \tag{74}$$

To get the freeze-out concentration of ^{7}Li we have to substitute in this formula the appropriate value of X_T, $K_1(T^*)$, and $X_{^3He}$ at the moment when ^{7}Li-freeze-out. This moment can be evaluated analyzing the freeze-out condition for ^{7}Li, ^{7}Be-reactions. If $X_D > 3 \times 10^{-5}(1 < \eta_{10} < 5)$ then freeze-out occurs after the deuterium gets its final abundance. Therefore to estimate $X_{^7Li}^f$ in this case one can substitute the above obtained values for $X_{^3He}^f$ and X_T^f in (74). If $\eta_{10} = 1$ the first term there dominates. Taking into account that $X_T^f \sim 0.01 \times X_D^f \simeq 4 \times 10^{-6}$ we get $X_{^7Li}^f(\eta_{10} = 1) \simeq 10^{-9}$. As η_{10} increases X_T^f drops and therefore $X_{^7Li}^f$ also decreases until the second term in (74) starts to dominate. After that the ^{7}Li abundance

starts to increase. This increase is due to two reasons. First of all it comes from the freeze-out temperature dependence of $K(T^*)$, which can be easily understood. Namely, the freeze-out temperature for ^{7}Li in this case is mostly determined by the efficiency of ^{7}Ben-reaction which in turn depends on the neutron concentration. In more dense universe the deuterium and free neutrons burned down more efficiently and disappear earlier (at higher temperature) than in the universe with small baryon density. Therefore the ^{7}Li concentration freezes-out at higher temperatures for which K_1 is bigger. Second reason is the following. If $\eta_{10} > 5$ the ^{7}Ben-reaction become inefficient before ^{3}He reaches its final freeze-out concentration. Therefore, to estimate $X^f_{^7\text{Li}}$ in this case we have to substitute in (74) the actual value of $X_{^3\text{He}}$ at the moment when the ^{7}Li-concentration freezes-out, which is bigger than $X^f_{^3\text{He}}$. The numerical calculation show that after passing through a relatively deep minimum $X^f_{^7\text{Li}}(\eta_{10} = 2 \div 3) \simeq 10^{-10}$, the litium concentration comes back to $\sim 10^{-9}$ at $\eta_{10} \simeq 10$. The "trough" in $X^f_{^7\text{Li}} - \eta_{10}$ dependence has a very simple explanation, namely, it is due to the competition of two reactions. In the universe with $\eta_{10} < 2 \div 3$ most of the litium-7 is produced directly as a result of ^{4}HeT-interactions. The efficiency of this process decreases with increase of η_{10} and as η_{10} becomes bigger than about $2 \div 3$, the reaction ^{7}Ben takes over compared to the direct ^{7}Li-production. In this case most of the litium-7 is produced via intermediate "^{7}Be-reservoir."

The berilium-7 is not so important from the observational point of view. Therefore, to get an idea about its expected concentration we estimate the amount of ^{7}Be only for $1 < \eta_{10} < 5$. In this case ^{7}Be freezes-out after deuterium and the quasi-equilibrium solution (61) for the free neutrons is still valid at this time. Hence we get

$$X^f_{^7\text{Be}} = \frac{7}{12} \frac{X_{^4\text{He}}}{\left(X^f_\text{D}\right)^2} R_1 \left(\frac{\lambda_{^3\text{He}^4\text{He}}}{\lambda_{^7\text{Be}n}}\right) X^f_{^3\text{He}} \sim O(1)10^{-12} \frac{X^f_{^3\text{He}}}{\left(X^f_\text{D}\right)^2}, \qquad (75)$$

where I used the experimental values for the ratios of the appropriate reactions; here the product of these ratios changes in about five times in the relevant temperature interval. At $\eta_{10} = 1$ we have $X^f_\text{D} \sim 4 \times 10^{-4}$, $X^f_{^3\text{He}} \sim 0.1 X^f_\text{D}$ and correspondingly $X^f_{^7\text{Be}} \sim 2.5 \times 10^{-10}$.

8. CONCLUSIONS

The above derived final abundances of the light elements are in very good agreement with the results of the numerical calculations reviewed for instance in Malaney and Mathews (1993), Olive *et al.* (2000), Sarkar (1996), Schramm and Turner (1998), and Walker *et al.* (1991). The analytical derivation is useful because it allows us to look inside "black box" (computer) and understand an interesting physics (for instance, attractors behavior etc.) taking place during nucleosynthesis.

Without referring to the computer codes one can estimate the final abundances of the light elements and understand their dependences on the main cosmological parameters. The theoretical calculations of the abundances are in agreement with observations of Malaney and Mathens (1993), Olive *et al.* (2000), Sarkar (1996), Schramm and Turner (1998), and Walker *et al.* (1991); and the estimates of the baryon density in the universe from nucleosynthesis come in impressive correspondence with CMB temperature fluctuation measurements. This gives strong support to the "standard cosmological model."

REFERENCES

Bennett C. L. *et al.* (2003). *First Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Maps and Basic Results*. astro-ph/0302207.

Berstein, J., Brawn, L. M., and Feinberg, G. (1989). *Review of Modern Physics* **61**, 25.

Burles, S., Nollett, K. M., and Turner, M. S. (1999). astro-ph/9903300.

Esmailzadeh, R., Starkman, G., and Dimopoulous, S. (1991). *Astrophysics Journal* **378**, 504.

Fields B. and Sarkar, S. (2002). Big-bang nucleosynthesis (in K. Hagiwara *et al.*). *Physical Review D* **66**, 010001.

Kawano, L. (1992). FERMILAB-PUB-92/04-A.

Kolb, E. and Turner, M. (1993). *The Early Universe*, Addison-Wesley, Boston, MA.

Malaney R. A. and Mathews, G. J. (1993). *Physics Report* **229**, 145.

Olive, K. A., Steigman, G., and Walker, T. P. (2000). *Physics Reports* **333**, 389.

Sarkar, S. (1996). *Report on Progress in Physics* **59**, 493.

Schramm D. N. and Turner, M. S. (1998). *Review of Modern Physics* **70**, 303.

Wagoner, R. V. (1973). *Astrophysics Journal* **179**, 343.

Wagoner, R. W., Fowler, W. A., and Hoyle, F. (1967). *Astrophysics Journal* **148**, 3.

Walker T. P. *et al.* (1991). *Astrophysics Journal* **376**, 51.

Weinberg, S. (1972). *Gravitation and Cosmology*, Wiley, New York.